Turbulent crossed fluxes in incompressible flows

Pedro Sancho
Instituto Nacional de Meteorología, Centro Zonal de Castilla y León, Orión 1, 47014 Valladolid, Spain
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We show in the framework of the stochastic calculus the existence of turbulent crossed fluxes in incompressible flows. Physically, these fluxes are related to the dependence of the phenomenological coefficients on the temperature and concentration variables.

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Two different approaches proposing the existence of turbulent crossed fluxes have been presented in the literature. In one of the approaches, which follows the thermodynamic analogy, these fluxes have been obtained for any type of flow whose energy distribution can be approximated by a Maxwell-type distribution [1]. In the other approach, based on the stochastic calculus, the effect is only present in compressible flows [2–5].

We shall show in this Brief Report that in the stochastic approach also it is possible to obtain turbulent crossed fluxes for incompressible flows. This result shows that the compressibility of the flow [2,3,5], the chemical reactions, and the phase transitions [4] are not the only physical mechanisms able to produce turbulent crossed fluxes.

In order to present the analysis in the simplest way we shall consider a mixture of gases advected by an incompressible turbulent flow. We suppose that the gases are chemically nonreacting and there are no phase transitions. The equations for the temperature T and number density n_i of admixtures are

$$\frac{\partial T}{\partial t}+\vec{v}\cdot\vec{\nabla}T=\kappa\Delta T \tag{1}$$

and

$$\frac{\partial n_i}{\partial t}+\vec{v}_i\cdot\vec{\nabla}n_i=\kappa_i\Delta n_i , \tag{2}$$

where $\vec{v}$ is the turbulent fluid velocity field, $\vec{v}_i$ the random velocity field of the admixtures, κ the coefficient of molecular thermal conductivity, and κ_i the coefficient of molecular diffusion of admixtures. In these equations, as usual on many occasions, we have neglected the effects associated with the terms $\vec{\nabla}\kappa\cdot\vec{\nabla}T$ and $\vec{\nabla}\kappa_i\cdot\vec{\nabla}n_i$, which we suppose are small.

The turbulent velocity of the gaseous admixture coincides with that of the surrounding fluid [4,5]. Then, if the turbulent velocity field of the fluid is incompressible, $\vec{\nabla}\cdot\vec{v}=0$, the turbulent velocity fields of the gaseous admixtures will be too, $\vec{\nabla}\cdot\vec{v}_i=0$.

A fundamental property of the phenomenological coefficients κ and κ_i is their dependence on the fundamental variables. As is well known the values of these coefficients vary with the temperature. Moreover, if we modify the concentrations of the admixtures the coefficients show important changes. Thus mathematically we have

$$\kappa=\kappa(T,n_j), \quad \kappa_i=\kappa_i(T,n_j). \tag{3}$$

Taking into account this property we are ready to show the existence of turbulent crossed fluxes in incompressible flows. The simplest mathematical form to derive this result is to consider perturbations of a reference state. This procedure is similar to the scenario considered in Ref. [4] to study the turbulent crossed effects in compressible flows with chemical reactions and phase transitions, in which a homogeneous equilibrium is considered as the reference state. We denote the values of the variables in the reference state by the superscript 0, i.e., T^0, etc. The next step is to study deviations from this state. Linearizing Eqs. (1) and (2) in the vicinity of the reference state we obtain the system of equations ruling the behavior of small perturbations. We introduce the notation $T=T^0+T^*$, $n_i=n_i^0+n_i^*$, $\vec{v}=\vec{v}^0+\vec{u}^*$, and $\vec{v}_i=\vec{v}_i^0+\vec{u}_i^*$. First, we need to consider the linearized form of Eq. (3):

$$\begin{aligned}\kappa(T,n_j)&\approx\kappa(T^0,n_j^0)+\left(\frac{\partial\kappa}{\partial T}\right)_0(T-T^0)+\sum_j\left(\frac{\partial\kappa}{\partial n_j}\right)_0(n_j-n_j^0)\\&=\kappa^0+\kappa_T^0T^*+\sum_j\kappa_j^0n_j^*\end{aligned} \tag{4}$$

and

$$\begin{aligned}\kappa_i(T,n_j)&\approx\kappa_i(T^0,n_j^0)+\left(\frac{\partial\kappa_i}{\partial T}\right)_0(T-T^0)\\&\quad+\sum_j\left(\frac{\partial\kappa_i}{\partial n_j}\right)_0(n_j-n_j^0)\\&=\kappa_i^0+\kappa_{iT}^0T^*+\sum_j\kappa_{ij}^0n_j^* ,\end{aligned} \tag{5}$$

where we have used an obvious notation.

Using these expressions, the equations for the perturbations of the reference state are

$$\frac{\partial T^*}{\partial t}+\vec{v}^0\cdot\vec{\nabla}T^*+\vec{u}^*\cdot\vec{\nabla}T^0=\kappa^0\Delta T^*+\kappa_T^*T^*+\sum_j\kappa_j^*n_j^* \tag{6}$$

and

$$\frac{\partial n_i^*}{\partial t}+\vec{v}_i^0\cdot\vec{\nabla}n_i^*+\vec{u}_i^*\cdot\vec{\nabla}n_i^0=\kappa_i^0\Delta n_i^*+\kappa_{iT}^*T^*+\sum_j \kappa_{ij}^*n_j^*, \tag{7}$$

where

$$\kappa_T^*=\kappa_T^0\Delta T^0,\quad \kappa_j^*=\kappa_j^0\Delta T^0,$$
$$\kappa_{iT}^*=\kappa_{iT}^0\Delta n_i^0,\quad \kappa_{ij}^*=\kappa_{ij}^0\Delta n_i^0. \tag{8}$$

In order to extract information from these equations we study the large-scale dynamics and average Eqs. (6) and (7) over an ensemble of random velocity fluctuations. A similar explicit calculation was presented in Ref. [4]. We note that, analytically, our Eqs. (6) and (7) are equivalent to Eq. (5) of Ref. [4]. Indeed, the terms $\kappa_T^*T^*+\Sigma_j\kappa_j^*n_j^*$ and $\kappa_{iT}^*T^*+\Sigma_j\kappa_{ij}^*n_j^*$ are equivalent to $P_\beta^\alpha a^\beta$. On the other hand, the terms $\vec{u}^*\cdot\vec{\nabla}T^0$ and $\vec{u}_i^*\cdot\vec{\nabla}n_i^0$ do not contain any dependence on T^* or n_i^*, just like the term Γ^α. Then Eq. (6) of Ref. [4] can be translated to our problem with obvious modifications:

$$\frac{\partial A^\alpha}{\partial t}+\vec{\nabla}\cdot(\vec{U}_\beta^\alpha A^\beta)=\vec{\nabla}\cdot(H_\beta^\alpha\vec{\nabla}A^\beta)+Q_\beta^\alpha A^\beta-\nabla\cdot\vec{J}^\alpha, \tag{9}$$

where $A^\alpha=\langle a^\alpha\rangle$ is the mean value of the variable $a^\alpha=(T^*,n_1^*,\ldots)$, $\vec{U}_\beta^\alpha=\vec{V}_\beta^\alpha+\sigma_*Q_\gamma^\alpha\vec{V}_\beta^\gamma$ where we use the notation $\vec{v}_\alpha^\beta=\vec{v}_\alpha\delta_{\alpha\beta}$ with $\vec{v}_i=\vec{V}_i+\vec{u}_i$, $\vec{V}_i=\langle\vec{v}_i\rangle$, and σ_* can be obtained as in Ref. [4], but with τ_c, the relaxation time related to chemical processes or phase transitions, replaced by an appropriate relaxation time for our problem. As explained in Ref. [4] this relaxation time can be obtained by calculating the trace of the tensor Q_β^α. The components of this tensor in our case are $Q_T^T=\kappa_T^*$, $Q_i^T=\kappa_i^*$, $Q_T^i=\kappa_{iT}^*$, $Q_j^i=\kappa_{ij}^*$, where we denote by T the label referring to the temperature and by i those corresponding to the different components. The rest of the terms in Eq. (9) are $H_\beta^\alpha=(\delta_\beta^\alpha+\mu Q_\gamma^\alpha)D_\beta^\gamma+\kappa_\beta^\alpha\delta$ with $\kappa_\beta^\alpha=\kappa_\alpha\delta_{\alpha\beta}$, $D_\beta^\alpha=\langle\tau\vec{u}_\gamma^\alpha\vec{u}_\beta^\gamma\rangle$, τ the momentum relaxation time of the random velocity field $\vec{u}$, and μ a parameter that can be calculated just like the parameter χ of Ref. [4] but with the change of the characteristic relaxation time of the process. Finally, we have $\vec{J}^\alpha=(-\langle\tau\vec{u}\vec{u}^*\cdot\vec{\nabla}T^0\rangle,-\langle\tau\vec{u}_1\vec{u}_1^*\cdot\vec{\nabla}n_1^0\rangle,\ldots)$.

Using all these expressions we obtain the following equations for the mean fields:

$$\frac{\partial\langle T^*\rangle}{\partial t}=\vec{\nabla}\cdot\left(B_T^T\vec{\nabla}\langle T^*\rangle+\sum_j B_j^T\vec{\nabla}N_j^*\right)+\phi_T \tag{10}$$

and

$$\frac{\partial N_i^*}{\partial t}=\vec{\nabla}\cdot\left(B_T^i\vec{\nabla}\langle T^*\rangle+\sum_j B_j^i\vec{\nabla}N_j^*\right)+\phi_i \tag{11}$$

with

$$B_T^T=(1+\mu\kappa_T^*)\langle\tau\vec{u}\vec{u}\rangle+\kappa,\quad B_i^T=\mu\kappa_i^*\langle\tau\vec{u}_i\vec{u}_i\rangle,$$
$$B_T^i=\mu\kappa_{iT}^*\langle\tau\vec{u}\vec{u}\rangle,\quad B_j^i=(\delta_{ij}+\mu\kappa_{ij}^*)\langle\tau\vec{u}_j\vec{u}_j\rangle+\kappa_i\delta_{ij}. \tag{12}$$

We have used the notation $N_i^*=\langle n_i^*\rangle$. The terms ϕ contain the second term in the left-hand side (lhs) and the second and third terms in the right-hand side (rhs) of Eq. (9). These terms represent physical processes that are not relevant for our purposes and will not be considered explicitly.

From Eqs. (10) and (11) we can easily derive the existence of turbulent crossed effects in this type of problems. In Eq. (10) the term $B_i^T\vec{\nabla}N_i^*$ can be interpreted as a turbulent analog of the Dufour effect; it describes the turbulent transport of heat due to the number density gradients of the admixtures. On the other hand, the term $B_T^i\vec{\nabla}\langle T^*\rangle$ is the turbulent analog of the Soret effect and represents the turbulent diffusion induced by the presence of thermal gradients.

This result shows the existence of turbulent crossed effects in the case of incompressible flows also and in the absence of chemical reactions and phase transitions. With this result we show that we can have turbulent crossed effects in any type of flow. We have also identified the physical mechanism behind this effect, namely the variation of the phenomenological coefficients with the temperature and the concentration of the mixtures.

Let us consider now the dependence of the strength of the effect on the different variables. To estimate the order of magnitude of the relevant terms we introduce the dimensionless form of the equations. Let us analyze, for instance, the crossed term in Eq. (10), $\partial\langle T^*\rangle/\partial t=\vec{\nabla}\cdot B_i^T\vec{\nabla}N_j^*+\cdots$, with $B_i^T=\mu(\partial\kappa/\partial n_i)_0\Delta T^0\langle\tau\vec{u}_i\vec{u}_i\rangle$ and $\mu=\tau_r[1-\tau_*-\tau_*^2\ln(\tau_*+\mathrm{Re}_*^{-s})]$ [4]. Now, τ_r is the relaxation time of the process considered in this work that replaces the relaxation time τ_c of the chemical reactions, $\tau_*=\tau_r/\tau_0$, and $\tau_0=l_0/u_0$ with l_0 the maximum scale of turbulent fluid motions and u_0 the characteristic turbulent fluid velocity in the scale l_0. Re_* is $\mathrm{Re}_*=\min(\mathrm{Re},\mathrm{Pe}_i)$ with $\mathrm{Re}=l_0u_0/\nu$ the Reynolds number, ν the kinematic viscosity, and $\mathrm{Pe}_i=l_0u_0/\kappa_i$ the Peclet number. Finally, we have $s=(q-1)/(3-q)$ with $q=2p-1$ and p the exponent of the spectrum of the kinetic turbulent energy of the fluid. By scaling length, velocity, time, temperature, and the phenomenological coefficients with the reference values l_0, u_0, l_0/u_0, $\kappa\nu/\alpha g l_0^3$ (with α the thermal expansion coefficient and g the acceleration of the gravity), and ν we obtain the dimensionless form of the equation:

$$\frac{\partial\langle T_a^*\rangle}{\partial t_a}=\mathrm{Re}^{-1}[1-\tau_*-\tau_*^2\ln(\tau_*+\mathrm{Re}_*^{-s})]$$
$$\times\vec{\nabla}_a\cdot\tau_r^a(\partial\kappa_a/\partial n_i)_0\Delta_aT_a^0\langle\tau_a\vec{u}_i^a\vec{u}_i^a\rangle\vec{\nabla}_aN_i^*+\cdots. \tag{13}$$

In this equation, a indicates that the variables are now dimensionless (note that τ_* is already dimensionless). The term Re^{-1} tends to diminish the intensity of the effect for high Reynolds number flows. However, the rest of the terms in the rhs of the first line of Eq. (13) (those depending on τ_*) tend to increase the intensity of the effect for large u_0. If we consider a problem whose geometric configuration is fixed (l_0 constant) and with τ_r approximately constant, the variation of Re and τ_* is given by u_0. For high Reynolds numbers, u_0 is very large, making τ_* also very large. In these conditions $\tau_*\mathrm{Re}^{-1}$ will be approximately constant, but

$\mathrm{Re}^{-1}\,\tau_*^2 \ln(\tau_* + \mathrm{Re}_*^{-s})$ will increase with the growth of u_0. Thus, in this particular situation, the strength of the effect increases for high Reynolds numbers. This conclusion agrees with the results obtained for the dependence of the turbulent Dufour effect on the Reynolds number derived in Ref. [1] (note that the ratio of increase of the effect with the Reynolds number obtained in Ref. [1] is different because of the different approach used for the evaluation). For other situations we must consider the variations of l_0 and/or τ_r, and the evaluation is much more difficult. Finally, note that the presence of a dependence of the intensity on $\ln(\mathrm{Re}_*)$ is typical of turbulent crossed fluxes [2–4].

The above considerations suggest some comments on the type of scenarios where crossed effects in incompressible flows could be relevant. These considerations show that when l_0 and τ_r can be considered approximately constant the intensity of the effect grows with the Reynolds number. Thus, very high Reynolds number flows are potential frameworks to observe the effects. However, other factors must be taken into account. As shown in Eq. (13) (and the equivalent equation for the Soret effect) the strength of the effect depends on $(\partial\kappa/\partial n_i)_0$, $(\partial\kappa_i/\partial T)_0$, ΔT^0, and Δn_i^0. When these variables are large the intensity of the effect increases. We must use mixtures for which the variation of the phenomenological coefficients with T and/or n_i is large. Moreover, we must consider experimental situations with large values of ΔT^0 and/or Δn_i^0. We can have large values of ΔT^0 in turbulent Rayleigh-Bénard convection with large external gradients of temperature. On the other hand, we can obtain large values of Δn_i^0 by placing a source of one of the components of the mixture in the flow. Other potentially interesting situations occur in the proximity of wall transitions, where strong variations of temperature and concentration are observed.

Finally, we note that in the case of the incompressible flows here considered a turbulent analog of the Onsager relations can be obtained, as in Ref. [5] for compressible flows. It is clear from the expressions for B_i^T and B_T^i that the turbulent crossed coefficients are given by $\langle \tau\vec{u}\vec{u}\rangle$ and $\langle \tau\vec{u}_i\vec{u}_i\rangle$, respectively, just as in the case of compressible flows [5]. Then, remembering as stressed at the beginning of this note that the turbulent velocity of the gaseous admixture coincides with that of the surrounding fluid, we have that the two turbulent crossed coefficients are equal; a turbulent analog of the Onsager relations is also valid for incompressible flows.

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[1] P. Sancho and J. E. Llebot, Phys. Lett. A **202**, 389 (1995).
[2] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. Lett. **76**, 224 (1996).
[3] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. E **55**, 2713 (1997).
[4] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. Lett. **80**, 69 (1998).
[5] P. Sancho, Phys. Rev. E **60**, 1762 (1999).